

Explicit integration of one problem of motion of the generalized Kowalevski top

M.P. Kharlamov*, A.Y. Savushkin

*Chair of Mathematical Simulation, Volgograd Academy of Public Administration,
Gagarin Street 8, Volgograd, 400131, Russia*

Abstract

In the problem of motion of the Kowalevski top in a double force field the 4-dimensional invariant submanifold of the phase space was pointed out by M.P. Kharlamov (Mekh. Tverd. Tela, **32**, 2002). We show that the equations of motion on this manifold can be separated by the appropriate change of variables, the new variables s_1, s_2 being elliptic functions of time. The natural phase variables (components of the angular velocity and the direction vectors of the forces with respect to the movable basis) are expressed via s_1, s_2 explicitly in elementary algebraic functions.

Key words: Kowalevski top, double force field, separation of variables, elliptic functions, explicit solution

MSC: 70E17, 70E40, 70H06

1 Introduction

The famous solution of S. Kowalevski [1] for the motion of a heavy rigid body about a fixed point was generalized for the case of double constant force field in [2,3]. This Hamiltonian system essentially has three degrees of freedom, and hardly can receive a clear geometrical or mechanical description of all types of motions. Invariant subsystems, which can be interpreted as systems with two degrees of freedom, were found in [2,4]. Phase topology of the case [2] was studied in [5]. The present paper deals with the case [4]. We show that by proper choice of local coordinates it is possible to obtain separated differential equations of motion, and express all phase variables explicitly in terms of two new variables, the latter being elliptic functions of time.

* Corresponding author.

Email address: mharlamov@vags.ru (M.P. Kharlamov).

2 Equations of motion and known first integrals

Consider a heavy magnetized rigid body with a fixed point placed in gravitational and magnetic constant force fields. Let $\vec{\alpha}, \vec{\beta}$ be the direction vectors of the force fields and \vec{e}_1, \vec{e}_2 be the radius vector of mass center and the vector of magnetic moment of the body. The scalar characteristics (for example, the product of weight and distance from the mass center to the fixed point) may be included in the length of either vector of the associated pair. We prefer to consider \vec{e}_1, \vec{e}_2 to be unit vectors according to the model accepted in [3]. The Euler–Poinsot equations of motion have the form

$$I \frac{d\vec{\omega}}{dt} = I\vec{\omega} \times \vec{\omega} + \vec{e}_1 \times \vec{\alpha} + \vec{e}_2 \times \vec{\beta}, \quad \frac{d\vec{\alpha}}{dt} = \vec{\alpha} \times \vec{\omega}, \quad \frac{d\vec{\beta}}{dt} = \vec{\beta} \times \vec{\omega}, \quad (1)$$

where $\vec{\omega}$ is the angular velocity, I is the inertia tensor. All vector or tensor objects are referred to the basis attached to the body.

Suppose that the body is dynamically symmetric and denote by π_e the equatorial plane of inertia ellipsoid. Choosing measure units one can always make the moment of inertia with respect to symmetry axis equal 1. Let

$$I = \text{diag}\{2, 2, 1\}, \quad (2)$$

$$\vec{e}_1, \vec{e}_2 \in \pi_e, \quad \vec{e}_1 \cdot \vec{e}_2 = 0.$$

Then \vec{e}_1, \vec{e}_2 may be taken as the first vectors of the movable basis. It is known that under conditions (2) the system (1) is completely integrable due to the existence of the first integrals [2]

$$H = \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - (\alpha_1 + \beta_2), \quad (3)$$

$$K = (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2,$$

and a new integral G pointed out in [3], which in case $\vec{\beta} = 0$ turns into the square of the vertical component of the angular momentum.

Below we exclude the case $|\vec{\alpha}| = |\vec{\beta}|$, $\vec{\alpha} \cdot \vec{\beta} = 0$, for which there exists a cyclic coordinate [6], and the problem reduces to the system with two degrees of freedom.

First, we show that without loss of generality one can always take $\vec{\alpha} \perp \vec{\beta}$.

The conditions (2) hold if we replace $\vec{e}_1, \vec{e}_2, \vec{\alpha}, \vec{\beta}$ with $\vec{e}_1(\theta), \vec{e}_2(\theta), \vec{\alpha}(\theta), \vec{\beta}(\theta)$,

where

$$\begin{aligned} (\vec{e}_1(\theta), \vec{e}_2(\theta)) &= (\vec{e}_1, \vec{e}_2)\Theta, \quad (\vec{\alpha}(\theta), \vec{\beta}(\theta)) = (\vec{\alpha}, \vec{\beta})\Theta, \\ \Theta &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta = \text{const.} \end{aligned} \quad (4)$$

Therefore, $\vec{e}_1(\theta), \vec{e}_2(\theta)$ may be taken as the first vectors of a new movable basis. At the same time, substitution (4) preserves the rotating moment $\vec{e}_1 \times \vec{\alpha} + \vec{e}_2 \times \vec{\beta}$ in Euler equations, and new vectors $\vec{\alpha}(\theta), \vec{\beta}(\theta)$ satisfy Poisson equations.

For the general case

$$(|\vec{\alpha}|^2 - |\vec{\beta}|^2)^2 + (\vec{\alpha} \cdot \vec{\beta})^2 \neq 0$$

take $\tan 2\theta = 2(\vec{\alpha} \cdot \vec{\beta})/(|\vec{\alpha}|^2 - |\vec{\beta}|^2)$ if $|\vec{\alpha}| \neq |\vec{\beta}|$, and $\cos 2\theta = 0$ if $|\vec{\alpha}| = |\vec{\beta}|$. Then $\vec{\alpha}(\theta) \perp \vec{\beta}(\theta)$. Thus, below we consider the natural restrictions (also called the geometrical integrals) in the form

$$|\vec{\alpha}|^2 = a^2, \quad |\vec{\beta}|^2 = b^2, \quad \vec{\alpha} \cdot \vec{\beta} = 0. \quad (5)$$

The first integral [3] can then be written in a simple way

$$G = \frac{1}{4}(\omega_\alpha^2 + \omega_\beta^2) + \frac{1}{2}\omega_3\omega_\gamma - b^2\alpha_1 - a^2\beta_2, \quad (6)$$

where

$$\omega_\alpha = 2\omega_1\alpha_1 + 2\omega_2\alpha_2 + \omega_3\alpha_3,$$

$$\omega_\beta = 2\omega_1\beta_1 + 2\omega_2\beta_2 + \omega_3\beta_3,$$

$$\omega_\gamma = 2\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + 2\omega_2(\alpha_3\beta_1 - \alpha_1\beta_3) + \omega_3(\alpha_1\beta_2 - \alpha_2\beta_1).$$

The conditions (5) make the phase space diffeomorphic to $M^6 = \mathbf{R}^3 \times SO(3)$. In general, the integral manifold $J_{h,k,g} = \{H = h, K = k, G = g\} \subset M^6$ consists of 3-dimensional tori bearing quasiperiodic trajectories dense on each torus for almost all values of the integral constants. Therefore, a 4-dimensional invariant submanifold, on which the induced system has a structure of the integrable system with two degrees of freedom, must reside in the set of critical points of the global integral mapping $J = H \times K \times G : M^6 \rightarrow \mathbf{R}^3$. One submanifold of this type was found in [2]:

$$M^4 = \{K = 0\} \subset M^6.$$

The topological structure of the induced system on M^4 was studied in [5].

Below we deal with the case [4], which generalizes the so-called marvellous motions of the 2nd and 3rd classes of Appelrot [7].

3 New equations for the integral manifolds

Changing, if necessary, the order of vectors in the movable basis we can assume that $a > b$. Denote

$$p^2 = a^2 + b^2, r^2 = a^2 - b^2$$

and consider the set $N^4 \subset M^6$ of critical points of the function

$$F = (2G - p^2 H)^2 - r^4 K$$

belonging to the level $F = 0$.

In order to obtain simple formulae and to establish the correspondence with [4], introduce new phase variables ($i^2 = -1$)

$$\begin{aligned} w_1 &= \omega_1 + i\omega_2, & w_2 &= \bar{w}_1, & w_3 &= \omega_3, \\ x_1 &= (\alpha_1 - \beta_2) + i(\alpha_2 + \beta_1), & x_2 &= \bar{x}_1, \\ y_1 &= (\alpha_1 + \beta_2) + i(\alpha_2 - \beta_1), & y_2 &= \bar{y}_1, \\ z_1 &= \alpha_3 + i\beta_3, & z_2 &= \bar{z}_1. \end{aligned} \tag{7}$$

The equations (1) can be then written as follows

$$\begin{aligned} 2w'_1 &= -(w_1 w_3 + z_1), & 2w'_2 &= w_2 w_3 + z_2, & 2w'_3 &= y_2 - y_1, \\ x'_1 &= -x_1 w_3 + z_1 w_1, & x'_2 &= x_2 w_3 - z_2 w_2, \\ y'_1 &= -y_1 w_3 + z_2 w_1, & y'_2 &= y_2 w_3 - z_1 w_2, \\ 2z'_1 &= x_1 w_2 - y_2 w_1, & 2z'_2 &= -x_2 w_1 + y_1 w_2. \end{aligned} \tag{8}$$

Here stroke stands for $d/d(it)$.

The conditions (5) take the form

$$\begin{aligned} z_1^2 + x_1 y_2 &= r^2, & z_2^2 + x_2 y_1 &= r^2, \\ x_1 x_2 + y_1 y_2 + 2z_1 z_2 &= 2p^2, \end{aligned} \tag{9}$$

and $F = 0, \nabla_6 F = 0$ give

$$F_1 = 0, \quad F_2 = 0, \tag{10}$$

where

$$\begin{aligned} F_1 &= \sqrt{x_1 x_2} w_3 - \frac{1}{\sqrt{x_1 x_2}} (x_2 z_1 w_1 + x_1 z_2 w_2), \\ F_2 &= \frac{i}{2} \left[\frac{x_2}{x_1} (w_1^2 + x_1) - \frac{x_1}{x_2} (w_2^2 + x_2) \right]. \end{aligned}$$

The equations (10) correspond to the system of invariant relations found in [4]. This fact, in particular, reveals the topological nature of strictly analytical results [4]. Moreover, it straightforwardly proves that N^4 is a subset of the phase space invariant under the flow of the dynamical system (1).

It is easy to see that almost everywhere on N^4 the system of equations (10) has rank 2, so N^4 has a natural structure of 4-dimensional manifold except, maybe, for a thin subset defined by $x_1 x_2 = 0$. Fix the constants h, k, g of the integrals (3), (6) and introduce new constants

$$m = \frac{1}{r^4}(2g - p^2 h), \quad \ell = \sqrt{2p^2 m^2 + 2hm + 1}$$

(the sign of ℓ is arbitrary). Then from the first integrals, conditions (9) and equations (10) we obtain on N^4

$$\begin{aligned} w_1^2 &= \frac{x_1}{x_2} r^2 m - x_1, & w_2^2 &= \frac{x_2}{x_1} r^2 m - x_2, \\ w_3 &= \frac{z_1 w_1}{x_1} + \frac{z_2 w_2}{x_2}, \\ m(x^2 + z^2) - \ell x + \sqrt{r^4 m^2 - r^2 m(x_1 + x_2) + x^2} &= 0, \end{aligned} \tag{11}$$

where

$$x^2 = x_1 x_2 \geq 0, \quad z^2 = z_1 z_2 \geq 0. \tag{12}$$

The square root in (11) equals $w_1 w_2$, and therefore is non-negative.

The equations (11) of the integral manifold $J_{h,k,g} \subset N^4$ show that in general case for given m, ℓ this manifold is two-dimensional.

4 Separation of variables

We now introduce new variables in (x, z) -plane

$$s_1 = \frac{x^2 + z^2 + r^2}{2x}, \quad s_2 = \frac{x^2 + z^2 - r^2}{2x}. \tag{13}$$

Calculating time derivatives from (8) we obtain

$$\begin{aligned} s_1' - s_2' &= \frac{r^2}{2x^2} [z_2 \sqrt{\frac{x_1}{x_2}} w_2 - z_1 \sqrt{\frac{x_2}{x_1}} w_1], \\ s_1' + s_2' &= \frac{r^2}{2x^2} [z_1 \sqrt{\frac{x_1}{x_2}} w_2 - z_2 \sqrt{\frac{x_2}{x_1}} w_1]. \end{aligned} \tag{14}$$

Let

$$\begin{aligned}\Psi(s_1, s_2) &= 4ms_1s_2 - 2\ell(s_1 + s_2) + \frac{1}{m}(\ell^2 - 1), \\ \Phi(s) &= 4ms^2 - 4\ell s + \frac{1}{m}(\ell^2 - 1).\end{aligned}$$

Then, having the obvious identity

$$\Psi^2(s_1, s_2) - \Phi(s_1)\Phi(s_2) = 4(s_1 - s_2)^2,$$

we find from (9), (11), (12)

$$\begin{aligned}x_1 &= -\frac{r^2}{2(s_1 - s_2)^2}[\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}], \\ x_2 &= -\frac{r^2}{2(s_1 - s_2)^2}[\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}], \\ y_1 &= 2\frac{(2s_1s_2 - p^2) - 2\sqrt{(s_1^2 - a^2)(s_2^2 - b^2)}}{\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ y_2 &= 2\frac{(2s_1s_2 - p^2) + 2\sqrt{(s_1^2 - a^2)(s_2^2 - b^2)}}{\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ z_1 &= \frac{r}{s_1 - s_2}(\sqrt{s_1^2 - a^2} + \sqrt{s_2^2 - b^2}), \\ z_2 &= \frac{r}{s_1 - s_2}(\sqrt{s_1^2 - a^2} - \sqrt{s_2^2 - b^2}), \\ w_1 &= r\frac{\sqrt{\Phi(s_2)} - \sqrt{\Phi(s_1)}}{\Psi(s_1, s_2) - \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ w_2 &= r\frac{\sqrt{\Phi(s_2)} + \sqrt{\Phi(s_1)}}{\Psi(s_1, s_2) + \sqrt{\Phi(s_1)\Phi(s_2)}}, \\ w_3 &= \frac{1}{s_1 - s_2}[\sqrt{(s_2^2 - b^2)\Phi(s_1)} - \sqrt{(s_1^2 - a^2)\Phi(s_2)}].\end{aligned}\tag{15}$$

Substitution of the latter expressions for $x_j, z_j, w_j (j = 1, 2)$ into (14) allows to obtain the differential equations for s_1, s_2 in the real form

$$\frac{ds_1}{dt} = \frac{1}{2}\sqrt{(a^2 - s_1^2)\Phi(s_1)}, \quad \frac{ds_2}{dt} = \frac{1}{2}\sqrt{(b^2 - s_2^2)\Phi(s_2)}.\tag{16}$$

Thus, s_1, s_2 are easily found as elliptic functions of time. The explicit formulae for the phase variables $\omega_j, \alpha_j, \beta_j (j = 1, 2, 3)$ immediately follow from (7), (15).

5 The types of solutions

The conditions (9) define the global area for the variables (13):

$$|s_1| \geq a, \quad |s_2| \leq b.$$

So, the domain of oscillations (16) for the fixed values of m, ℓ is obtained from the inequalities

$$\Phi(s_1) \leq 0, \quad \Phi(s_2) \geq 0.$$

Bifurcations of solutions (16) with respect to the parameters m, ℓ take place when $\Phi(\pm a) = 0$, or $\Phi(\pm b) = 0$. It leads to a set of lines in (m, ℓ) -plane

$$\ell = -2am \pm 1, \quad \ell = 2am \pm 1, \quad \ell = -2bm \pm 1, \quad \ell = 2bm \pm 1.$$

Analyzing the evolution of roots of the polynomial $\Phi(s)$, we obtain all different types of motions.

The critical motions appear in the cases when one of the variables s_1, s_2 remains constant, coinciding with the double root of polynomial product in the right part of the corresponding equation (16). It obviously leads to the motion of the body of pendulum type: either $\omega_1 = \omega_3 \equiv 0$, or $\omega_2 = \omega_3 \equiv 0$.

An interesting aspect of the case considered is that both movable and immovable hodographs of the angular velocity are explicitly found simultaneously without any further integration. Actually, because for almost all constants m, ℓ from the domain of existing of real solutions, hodographs fill some two-dimensional surfaces densely, we use the expressions (15) to obtain the parametric equations of those surfaces, in which s_1, s_2 are independent parameters. Expressions for $\vec{\alpha}, \vec{\beta}$ (and, therefore, for $\vec{\alpha} \times \vec{\beta}$) via s_1, s_2 give the parametric equations for the orientation matrix. Contemporary methods of computer graphics provide the possibility to construct a detailed and clear picture of motion as rolling without slipping of one surface through the other: at any moment these surfaces have a common point with zero absolute velocity and one common tangent vector, expressing the fact that absolute and relative derivatives of the angular velocity coincide.

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